

One last example in Spherical Coordinates

Ex: Compute the volume of a disk of radius $\alpha > 0$

NB: We already did this in Cartesian coordinates but it was nasty!!

Solution: In spherical coordinates $D_\alpha = \{(p, \theta, \phi) : 0 \leq p \leq \alpha, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$

$$\begin{aligned} \text{Vol}(D_\alpha) &= \iiint_{D_\alpha} 1 \, dV_{\text{cart}} & dV_{\text{cart}} &= p^2 \sin(\phi) \, dV_{\text{sph}} \\ &= \iiint_{D_\alpha} 1 \cdot p^2 \sin(\phi) \, dV_{\text{sph}} = \int_{p=0}^{\alpha} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} p^2 \sin(\phi) \, d\phi \, d\theta \, dp \\ &= \int_{p=0}^{\alpha} \int_{\theta=0}^{2\pi} -p^2 [\cos(\phi)]_{\phi=0}^{\pi} \, d\theta \, dp = \int_{p=0}^{\alpha} \int_{\theta=0}^{2\pi} -p^2 (-1-1) \, d\theta \, dp \\ &= 2 \int_{p=0}^{\alpha} p^2 [\theta]_{\theta=0}^{2\pi} \, dp = 2 \int_{p=0}^{\alpha} (2\pi) \cdot p^2 \, dp = 4\pi \left(\frac{1}{3} p^3 \right)_{p=0}^{\alpha} \\ &= \frac{4}{3} \pi \alpha^3 \end{aligned}$$

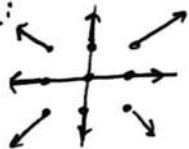
§16.2 : Vector Fields

Goal: Study $\vec{v} : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Defn: A vector field on \mathbb{R}^n is a function $\vec{v} : \mathbb{R}^n \rightarrow \mathbb{R}^m$
(vector field may be abbreviated as v.f.)

Ex: $\vec{v}(x, y) = \langle x, y \rangle$ is a v.f. on \mathbb{R}^2

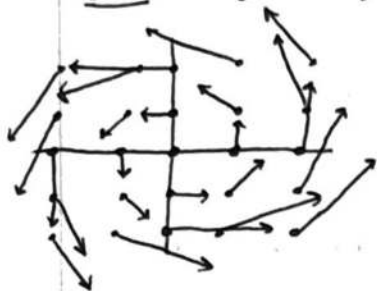
Picture:



$$\vec{v}(1, 0) = \langle 1, 0 \rangle \rightarrow$$

NB: in pictures, we shift $\vec{v}(x, y)$ to have tail (x, y)

Ex 2: draw $\vec{v}(x,y) = \langle -y, x \rangle$



$$v(0,0) = \langle 0,0 \rangle$$

$$v(1,1) = \langle -1,1 \rangle$$

$$v(1,0) = \langle 0,1 \rangle$$

$$v(1,2) = \langle -2,1 \rangle$$

$$v(1,-2) = \langle 2,1 \rangle$$

Ex: Given any function $F: \mathbb{R}^n \rightarrow \mathbb{R}$, we obtain a vector field by taking the gradient:

e.g. $F(x,y) = xy$

$\nabla F(x,y) = \langle y, x \rangle$ is the gradient vector field of f .

e.g. $f(x,y,z) = e^{x+y^2} \cos(x+z)$

$$\nabla F(x,y,z) = \langle e^{x+y^2} \cos(x+z) - e^{x+y^2} \sin(x+z), 2ye^{x+y^2} \cos(x+z) - e^{x+y^2} \sin(x+z) \rangle$$

is a vector field on \mathbb{R}^3

e.g. $F(x,y) = x^2 + 3xy - xy^2$

$$\nabla F(x,y) = \langle 2x + 3y - y^2, 3x - 2xy \rangle$$

Terminology: ① A vector field is conservative when it is the gradient vector field of some function.

② When $\vec{v} = \nabla F$ is conservative we say f is a potential function for \vec{v} .

Obvious Question: Which v.f.s are conservative? are all v.f.s conservative

If $\vec{v}(x,y)$ is conservative, then $\vec{v} = \nabla F(x,y)$

i.e. $\vec{v}(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

By Clairaut's Theorem, $f_{xy} = f_{yx}$, so for $\vec{v} = \langle v_x, v_y \rangle$ we have

$$\frac{\partial}{\partial y}[v_x] = \frac{\partial}{\partial x}[v_y] \text{ for all conservative v.f.s}$$

Ex: $\vec{v} = \langle -y, x \rangle$ is not conservative! Because...

$$\frac{\partial}{\partial x}[V_y] = \frac{\partial}{\partial x}[x] = 1 \neq -1 = \frac{\partial}{\partial y}[-y] = \frac{\partial}{\partial y}[V_x]$$

It turns out this is an "iff" type condition!

Proposition: A vector field $\vec{v}(x_1, x_2, \dots, x_n) = \langle v_1, v_2, \dots, v_n \rangle$ is conservative if and only if for all i, j we have $\frac{\partial}{\partial x_i}[v_j] = \frac{\partial}{\partial x_j}[v_i]$ (i.e.) a v.f. is conservative iff it satisfies (Curlant's Theorem)

NB A proof of this ~~result~~ result follows from the methods I give below

Ex: $\vec{v} = \langle \overset{v_x}{x}, \overset{v_y}{y} \rangle$ Conservative? If yes, potential

Sol: $\frac{\partial}{\partial x}[v_y] = \frac{\partial}{\partial x}[y] = 0$

To compute the potential: If $\vec{v} = \nabla f$, then

$$\frac{\partial}{\partial y}[v_x] = \frac{\partial}{\partial y}[x] = 0$$

$$f_x(x, y) = x \text{ and } f_y(x, y) = y$$

$$\therefore f(x, y) = \int \frac{df}{dx} dx = \int x dx = \frac{1}{2}x^2 + C(y)$$

$$\therefore y = f_y(x, y) = \frac{\partial}{\partial y}[\frac{1}{2}x^2 + C(y)] = \frac{\partial C}{\partial y}$$

↪ honest constant

$$\text{Hence } C(y) = \int \frac{\partial C}{\partial y} dy = \int y dy = \frac{1}{2}y^2 + D$$

$$\therefore f(x, y) = \frac{1}{2}x^2 + C(y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + D \text{ is a potential for } \vec{v} \text{ for every constant } D \quad \square$$

Ex: $\vec{v} = \langle 2xy, x^2 - 3y^2 \rangle$ Conservative? If so, compute potential

Sol: $\frac{\partial}{\partial x}[v_y] = \frac{\partial}{\partial x}[x^2 - 3y^2] = 2x$

$\therefore \vec{v}$ is conservative, i.e. $\vec{v} = \nabla f$ for

$$\frac{\partial}{\partial y}[v_x] = \frac{\partial}{\partial y}[2xy] = 2x$$

some $f(x, y)$. $\frac{df}{dx} = 2xy$ and $\frac{df}{dy} = x^2 - 3y^2$

$$\therefore f(x, y) = \int \frac{df}{dx} dx = \int 2xy dx = x^2y + C(y)$$

$$\text{Hence } x^2 - 3y^2 = \frac{df}{dy} = \frac{\partial}{\partial y}[x^2y + C(y)] = x^2 + \frac{\partial C}{\partial y}$$

↪ (honest constant)

$$\therefore \frac{\partial C}{\partial y} = -3y^2 \quad \text{so } C(y) = \int \frac{\partial C}{\partial y} dy = \int -3y^2 dy = -y^3 + D$$

$$\therefore f(x, y) = x^2y + C(y) = x^2y - y^3 + D \text{ is a potential function for } \vec{v} \text{ for every constant } D \quad \square$$

Ex $v(x,y) = \langle \frac{1}{x+y}, \ln(x+y) \rangle$ Conservative? Potential?

Sol: $\frac{d}{dx}[\ln(x+y)] = \frac{1}{x+y}$ and $\frac{d}{dy}[(x+y)^{-1}] = -(x+y)^{-2}$

$\therefore \frac{\partial v_x}{\partial x} \neq \frac{\partial v_y}{\partial y}$ so v is not conservative \square